Rotating and stratified fluid flow

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For flows in either rotating or stratified fluids, a technique is developed for solving initial-value problems using an Oseen approximation to the non-linear inertial terms in the equations of motion. The resulting equations for either application are similar. The solutions bear a strong qualitative resemblence to observed flows of both kinds, being characterized at small Rossby or Froude numbers by a blocked flow upstream of an obstacle and waves on the downstream side.

1. Introduction

The well-known similarity between the experimental observations of the flow of a rotating fluid and a stratified fluid past an obstacle, led the author to investigate whether the theory developed for rotating fluids could be extended to stratified fluids. The important features of both types of flow are the blocking of the upstream flow and the existence of internal waves on the downstream side of the obstacle for low Rossby and Froude numbers (see Long 1953*a*, *b*).

So far the work in rotating fluids has followed one of two approaches. Proudman (1916), Grace (1926), Morgan (1951) and Stewartson (1952) used the linearized non-steady equations of motion to solve initial-value problems. Proudman and Grace looked for a series solution, whereas Morgan and Stewartson used the more successful Laplace-transform technique. Morgan and Stewartson showed that the motion in the limit of infinite time is two-dimensional everywhere except on a certain singular surface (the circumscribing cylinder) and the axis of rotation, where it is non-steady.

Taylor (1922), Long (1953*a*) and Fraenkel (1956) based their work on the steady non-linear equations of motion, which reduce to a linear equation for the stream function, if uniform upstream conditions are assumed. Their solutions describe a wave-like motion for low Rossby numbers and an irrotational-like flow for higher Rossby numbers. The disadvantages of this theory are the need to justify the neglect of upstream waves, which are a possible solution, and the lack of any terms describing a geostrophic flow.

The main work in stratified fluid flow (Long 1953*b*, 1959; Yih 1958, 1959, 1960) has used the steady non-linear equations of motion, which, again with the assumption of suitable uniform upstream conditions, reduce to a linear equation. The solutions obtained are similar to those obtained for rotating fluids and suffer from the same disadvantages.

Yih (1959) showed that for steady weak motions the effect of gravity is to inhibit vertical motions and horizontal density gradients. He supported his conclusions by a simple experiment in which a paddle was moved slowly through a stratified fluid. A little stratification was found sufficient to make the effect of the paddle felt far upstream and downstream.

In the work which follows an attempt is made to combine the two approaches by using the non-steady equations of motion with an Oseen-type approximation to the non-linear inertia terms. Initial-value problems are solved by the Laplace-transform technique and the steady-state solutions are obtained by taking the limiting case of infinite time. These solutions contain the type of solution derived by Morgan (1951) and Stewartson (1952) as a special case. For given problems in both rotating and stratified fluids, parts of the solutions are the same as those obtained by Long, Fraenkel and Yih, i.e. irrotational-like terms for high Rossby and Froude numbers plus wave-like terms for smaller Rossby and Froude numbers. In addition, the solutions in this paper contain terms which describe geostrophic or one-dimensional flow, i.e. blocked flow. One of the results suggests that the geostrophic terms depend on the way the steady-state is achieved, whereas the irrotational and wave-like terms are only dependent on the Rossby or Froude number and the geometry of the problem. In the case of rotating fluids the effects of the introduction and neglect of viscosity are also discussed. Probably the main value of the work is that it helps to co-ordinate the apparently incompatible results of previous workers.

2. The equations of motion of a viscous rotating fluid

In this section we consider the problem of an infinite domain of viscous incompressible fluid which is initially rotating with angular velocity Ω about Ox and has uniform axial velocity U along Ox. At the time $t = 0$ a small axisymmetric perturbation is introduced and maintained on the plane $x = 0$. The problem is to find the nature of the flow as $t \to \infty$.

The equations of motion of a viscous incompressible fluid referred to axes rotating with angular velocity $\mathbf{\Omega}$ are (see Squire 1956, p. 140)

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\,.\,\nabla)\,\mathbf{v} + 2\mathbf{\Omega}\wedge\mathbf{v} = -\nabla P + \nu\nabla^2\mathbf{v}, \tag{1}$$

with $P = \rho^{-1}p - \Phi - \frac{1}{2}\Omega^2 d^2$, where p is the pressure, ρ the density, Φ the potential of the body force, d is the distance from the axis of rotation and ν is the kinematic viscosity. The velocity components parallel to the (r, ϕ, x) directions are $(u, v, w+U)$ respectively, where u, v and w are assumed to be small compared with U. Since the perturbation is axisymmetric, $\partial/\partial\phi \equiv 0$. Using this condition and neglecting terms of $O(u^2)$, the equations of motion (1) reduce to

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)(u, v, w) + 2\Omega(-v, u, 0) = \left(-\frac{\partial P}{\partial r}, 0, -\frac{\partial P}{\partial x}\right) + \nu\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{\partial^2}{\partial x^2}\right)(u, v, w) - \frac{\nu}{r^2}(u, v, 0), \tag{2}$$

and the equation of continuity becomes

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial x} = 0. \tag{3}$$

From (3) we can define a Stokes stream function ψ of the perturbed flow by

$$ru = -\frac{\partial \psi}{\partial x}, \quad rw = \frac{\partial \psi}{\partial r}. \tag{4}$$

Eliminating P and v from (2) and using (4), the following equation is obtained for ψ,

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} - \nu D^2\right)^2 D^2\psi + 4\Omega^2 \frac{\partial^2 \psi}{\partial x^2} = 0, \tag{5}$$

where
$$D^2 = \left(r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial x^2}\right).$$

Initial-value problems can usually be solved by the Laplace-transform method and we define the Laplace transform by

$$\bar{f}(p) = \int_0^\infty e^{-pt} f(t)\, dt. \tag{6}$$

The Laplace transform of (5) is

$$\left(p + U\frac{\partial}{\partial x} - \nu D^2\right)^2 D^2\bar{\psi} + 4\Omega^2 \frac{\partial^2 \bar{\psi}}{\partial x^2} = 0, \tag{7}$$

provided $D^2\psi = 0$ at $t = 0$ and $D^2 \partial\psi/\partial t = 0$ at $t = 0$. The first condition is satisfied provided the viscous stresses remain finite, as the initial perturbation flow is irrotational (see Lamb 1932, p. 11); and by eliminating P from (2) we can see that the second condition is identically satisfied, as the azimuthal component of velocity v is initially zero everywhere.

Equation (7) admits solutions of the form

$$\bar{\psi} = rJ_1(kr)\exp(-\alpha k x) \tag{8}$$

where α is a root of the sextic

$$[p - U\alpha k - \nu k^2(\alpha^2 - 1)]^2 (\alpha^2 - 1) + 4\Omega^2\alpha^2 = 0, \tag{9}$$

and $J_1(kr)$ is the Bessel function of order one. Since the perturbation imposed on the plane $x = 0$ can be written in the form

$$\psi(r, 0, t) = \int_0^\infty A(k, t)\, rJ_1(kr)\, dk,$$

it is sufficient to discuss the behaviour of one Bessel component.

The boundary conditions at infinity on the perturbation stream function ψ are

$$\psi \to 0 \quad \text{as} \quad x \to \pm\infty \quad \text{for} \quad \text{fixed } t,$$

which imply
$$\bar{\psi} \to 0 \quad \text{as} \quad x \to \pm\infty \quad \text{for} \quad \mathrm{Re}(p) > 0,$$

where $\mathrm{Re}(p)$ denotes the real part of p. Hence the roots of (9) describing the downstream flow ($x > 0$) must satisfy

$$\mathrm{Re}(\alpha) > 0 \quad \text{for} \quad \mathrm{Re}(p) > 0, \tag{10}$$

and those describing the upstream flow ($x < 0$) must satisfy

$$\mathrm{Re}(\alpha) < 0 \quad \text{for} \quad \mathrm{Re}(p) > 0. \tag{11}$$

It is convenient to introduce the following non-dimensional variables and constants to discuss the roots of (9):

$$s = p/2\Omega, \quad R_0 = Uk/2\Omega, \quad R = 2\Omega/\nu k^2,$$

where R_0 is a Rossby number based on the wave-number k of the perturbation and R is a Reynolds number estimating the ratio of the Coriolis force to the viscous force. Equation (9) now takes the form

$$(s - R_0\alpha - R^{-1}(\alpha^2 - 1))^2(\alpha^2 - 1) + \alpha^2 = 0. \tag{12}$$

Case	Upstream (Re$(\alpha) < 0$ for Re$(s) > 0$)	Downstream (Re$(\alpha) > 0$ for Re$(s) > 0$)
(i) $R^{-1} = 0$, $R_0 = 0$. Morgan (1951), Stewartson (1952)	$\alpha \sim -s$, geostrophic	$\alpha \sim s$, geostrophic
(ii) R^{-1} small, $R_0 = 0$ Morrison & Morgan (1956)	$\alpha \sim -R^{-1}$, geostrophic-like; $\alpha \sim -(\frac{1}{2}R)^{\frac{1}{2}}(1 \pm i)$, Eckman spiral	$\alpha \sim R^{-1}$, geostrophic-like; $\alpha \sim (\frac{1}{2}R)^{\frac{1}{2}}(1 \pm i)$, Eckman spiral
(iii) $R^{-1} = 0$, $0 < R_0 < 1$. Stewartson (1958)	$\alpha \sim -s(1-R_0)^{-1}$, geostrophic	$\alpha \sim s(1+R_0)^{-1}$, geostrophic; $\alpha \sim \pm i(R_0^{-2} - 1) + s(R_0 - R_0^3)^{-1}$, wave
$R^{-1} = 0$, $1 < R_0$	$\alpha = -(1 - R_0^{-2})^{\frac{1}{2}}$, irrotational-like	$\alpha \sim s(R_0 \pm 1)^{-1}$, geostrophic; $\alpha = (1 - R_0^{-2})^{\frac{1}{2}}$, irrotational-like
(iv) $R^{-1} \neq 0$, $0 < R_0 < 1$, $RR_0 \gg 1$	$\alpha \sim -\{R(1-R_0)\}^{-1}$ geostrophic-like; $\alpha \sim -RR_0$ (twice), suction boundary layer	$\alpha \sim \{R(1+R_0)\}^{-1}$, geostrophic-like; $\alpha \sim \pm i(R_0^{-2} - 1)^{\frac{1}{2}} + \{RR_0^3(1 - R_0^2)\}^{-1}$, wave-like
$R^{-1} \neq 0$, $1 < R_0$, $RR_0 \gg 1$	$\alpha \sim -(1 - R_0^{-2})^{\frac{1}{2}}$, irrotational-like; $\alpha \sim -RR_0$ (twice), suction boundary layer	$\alpha \sim (1 - R_0^{-2})^{\frac{1}{2}}$, irrotational-like; $\alpha \sim \{R(R_0 \pm 1)\}^{-1}$, geostrophic-like

TABLE 1. Asymptotic form of roots α as $s \to 0$ or at $s = 0$.

From (12) it is easily seen that the locus of imaginary α lies in the half-plane Re$(s) \leqslant 0$ and only coincides with the imaginary axis for $R^{-1} = 0$. Hence the Re(α) has the same sign for Re$(s) > 0$, provided the α's are defined as single-valued functions of s with the branch cuts extending to Re$(s) = -\infty$.

Now the values of α at $s = 0$ correspond to the steady-state solution and the roots describing the upstream and downstream flows are determined by satisfying the boundary conditions at upstream and downstream infinity. For $R^{-1} \neq 0$ the Re$(\alpha) \neq 0$ at $s = 0$ and the flow is determined, but for $R^{-1} = 0$ the roots α can be imaginary or zero at $s = 0$. This leads to the type of indeterminacy experienced in some steady inviscid flow problems, e.g. surface waves, waves in rotating fluids and in incompressible stratified fluids (see Long 1953*a*, *b*), where waves can theoretically occur upstream as well as downstream of an obstacle but are not observed in practice. The indeterminacy can be removed either by including viscosity or by solving the initial-value problem.

The characteristics of the flow for large values of t are discussed in table 1 above, using the asymptotic form of the roots α for small s.

In the work that follows only case (iii) in table 1 will be considered for which (7) reduces to

$$\left(p+U\frac{\partial}{\partial x}\right)^2 D^2\overline{\psi}+4\Omega^2\frac{\partial^2\overline{\psi}}{\partial x^2} = 0, \tag{13}$$

which has solutions finite at $r = 0$ of the form

$$\overline{\psi} = rJ_1(kr)\exp\left(-\alpha kx\right), \tag{14}$$

where α is a root of

$$(s-R_0\alpha)^2(\alpha^2-1)+\alpha^2 = 0,\quad R_0 = \frac{Uk}{2\Omega},\quad s = \frac{p}{2\Omega}. \tag{15}$$

3. The equations of motion of a stratified fluid

The study is restricted to two-dimensional flows of a stratified fluid, which is assumed to be incompressible, inviscid and non-diffusive. The Euler equations for such flows are

$$\left.\begin{aligned}\rho\frac{\partial u}{\partial t}+\rho u\frac{\partial u}{\partial x}+\rho w\frac{\partial u}{\partial z} &= -\frac{\partial p}{\partial x},\\ \rho\frac{\partial w}{\partial t}+\rho u\frac{\partial w}{\partial x}+\rho w\frac{\partial w}{\partial z} &= -\rho g-\frac{\partial p}{\partial z},\end{aligned}\right\} \tag{16}$$

in which u, w are the velocity components parallel to Ox and Oz respectively, where z is measured in a direction opposing gravity. Since the fluid is incompressible and non-diffusive

$$\frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial x}+w\frac{\partial\rho}{\partial z} = 0, \tag{17}$$

and the continuity equation reduces to

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z} = 0. \tag{18}$$

We now derive the equations of motion appropriate to the following type of initial-value problem. The flow initially consists of a unidirectional flow in the x-direction with constant velocity U and density stratification $\rho_0-\beta z$ between horizontal planes $z = 0$ and $z = d$, where $\beta > 0$ is a constant. At the time $t = 0$ a perturbation is introduced on the plane $x = 0$. Let the subsequent velocity components parallel to the x and z axes be $(u+U, w)$, the density $\rho_0-\beta z+\rho$, and the pressure p_0+p, where u and w, ρ, p are assumed to be small compared with $U, \rho_0-\beta z, p_0$, respectively and $dp_0/dz = -(\rho_0-\beta z)g$. Since the initial density stratification is small we assume that the only effect of variation in density is the generation of buoyancy forces, and that the effects of variation in density on the inertia terms can be neglected. Using this, the Boussinesq approximation, and taking terms of $O(u, w, \rho, p)$, (16), (17) and (18) reduce to

$$\left.\begin{aligned}\rho_0 \partial u/\partial t + \rho_0 U \partial u/\partial x &= -\partial p/\partial x,\\ \rho_0 \partial w/\partial t + \rho_0 U \partial w/\partial x &= -\rho g - \partial p/\partial z,\end{aligned}\right\} \tag{19}$$

$$\partial \rho/\partial t + U \partial \rho/\partial x - w\beta = 0, \tag{20}$$

$$\partial u/\partial x + \partial w/\partial z = 0. \tag{21}$$

From (21) we define a perturbation stream function ψ by

$$u = \partial\psi/\partial z, \quad w = -\partial\psi/\partial x. \tag{22}$$

On eliminating p from (19) and using (22) we obtain

$$\rho_0(\partial/\partial t + U \partial/\partial x)\nabla^2\psi = g\,\partial\rho/\partial x, \quad \nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial z^2, \tag{23}$$

and from (23) and (20) we have

$$(\partial/\partial t + U\partial/\partial x)^2\nabla^2\psi + \hat{g}\,\partial^2\psi/\partial x^2 = 0, \quad \hat{g} = g\beta/\rho_0. \tag{24}$$

Since the perturbation is introduced at time $t = 0$, we can define a Laplace transform by (6) and derive the following equation for $\bar{\psi}$

$$(p + U\partial/\partial x)^2\nabla^2\bar{\psi} + \hat{g}\,\partial^2\bar{\psi}/\partial x^2 = 0, \tag{25}$$

on using the condition that the initial perturbed motion is irrotational and that the initial density perturbation is zero (see (23)), which imply that

$$\nabla^2\psi = 0, \quad \nabla^2\partial\psi/\partial t = 0 \quad \text{at} \quad t = 0.$$

Equation (25) admits solutions of the form

$$\bar{\psi} = (A\cos kz + B\sin kz)\exp(-\alpha k x) \tag{26}$$

where α is a root of

$$(p - U\alpha k)^2(\alpha^2 - 1) + \hat{g}\alpha^2 = 0$$

or on putting

$$s = \hat{g}^{-\frac{1}{2}}p, \quad R_i = \hat{g}^{-\frac{1}{2}}Uk. \tag{27}$$

where R_i is a Froude number which estimates the magnitude of the inertia forces to the buoyancy forces, the above equation for α reduces to

$$(s - R_i\alpha)^2(\alpha^2 - 1) + \alpha^2 = 0. \tag{28}$$

On comparing (28) with (15) we see that the equation for α is identical except that the Froude number R_i replaces the Rossby number R_0. The solution (26) is similar to (14). Since trigonometric functions are easier to handle than Bessel functions, the theory will be developed for a stratified fluid and the solutions obtained adjusted for a rotating fluid.

4. Definition of $\alpha_i(s)$ as a single-valued function of s

Before a solution for ψ can be obtained it is necessary to examine the nature of the singularities of $\alpha(s)$ in the complex s-plane, so that the Laplace inversion integral can be evaluated. The branch points of $\alpha(s)$ will occur at values of s where two or more of the roots are coincident. The values of α and s at the branch points are given by (28) and the derivative of (28) with respect to α, which give

$$(s-R_i\alpha)(s-R_i\alpha^3) = 0.$$

Provided $R_i \neq 1$, the condition $R_i\alpha = s$ leads to two coincident roots $\alpha = 0$ at $s = 0$ and since $\alpha \sim s(R_i \pm 1)^{-1}$ near $s = 0$, it follows that the origin is not a

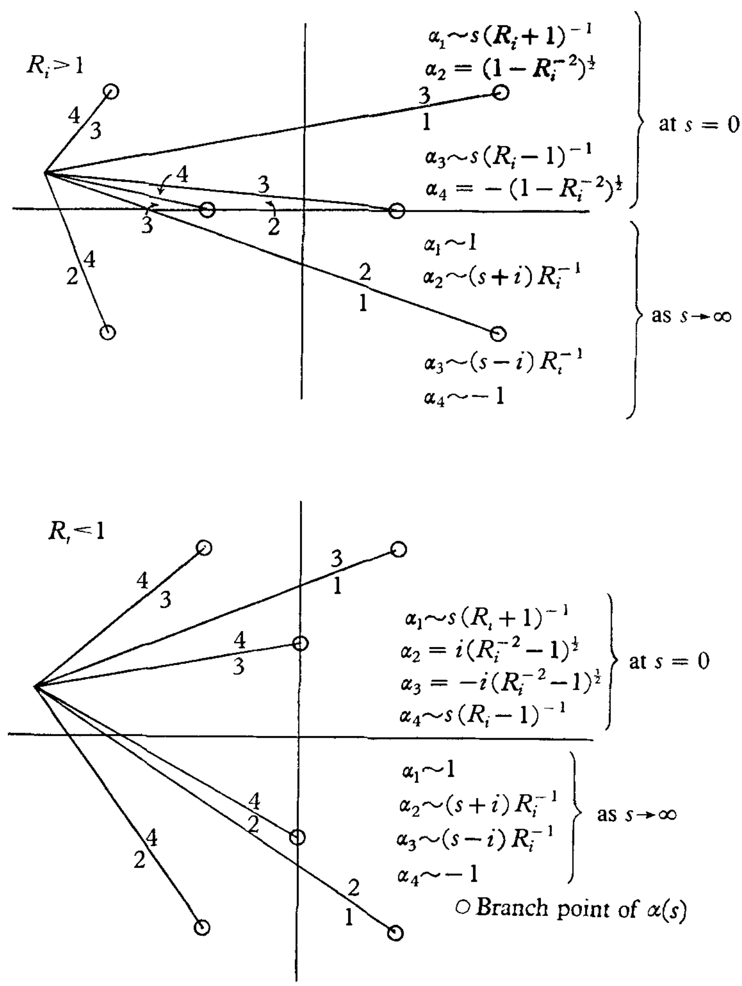

FIGURE 1. Definition of $\alpha(s)$.

branch point. The other condition $R_i\alpha^3 = s$ leads to branch points of two of the α's at the following values of s

$$s = \pm R_i(1-R_i^{-\frac{2}{3}})^{\frac{3}{2}}, \quad \pm R_i(1-\omega R_i^{-\frac{2}{3}})^{\frac{3}{2}}, \quad \pm R_i(1-\omega^2 R_i^{-\frac{2}{3}})^{\frac{3}{2}},$$

where $(1, \omega, \omega^2)$ are the cube roots of unity. It is worth noting that $s = \infty$ is not a branch point.

If $R_i = 1$ the first two branch points coincide with the origin and all the roots are zero there. Near $s = 0$ the roots behave like

$$\alpha \sim \tfrac{1}{2}s, \quad -(2s)^{\frac{1}{3}}, \quad -\omega(2s)^{\frac{1}{3}}, \quad -\omega^2(2s)^{\frac{1}{3}}.$$

We now define the roots $\alpha(s)$ as single-valued functions of s by assigning at $s = \infty$, $\alpha_1 = 1$ to sheet 1, $\alpha_2 \sim (s+i)/R_i$ to sheet 2, $\alpha_3 \sim (s-i)/R_i$ to sheet 3 and $\alpha_4 = -1$ to sheet 4. By considering the variation of the roots along the real and imaginary axes, we can determine where the sheets hang together and hence define $\alpha(s)$ in the cases $R_i > 1$ and $R_i < 1$ as shown in figure 1.

5. The solution for one Fourier component

Let the perturbation introduced on the plane $x = 0$ in a uniform flow of stratified fluid be

$$\psi(0,z,t) = (u_0/k)\sin kz\, H(t),$$

where u_0 is a constant and $H(t)$ is the Heaviside function defined by $H(t) = 0$ for $t < 0$ and $H(t) = 1$ for $t \geqslant 0$. The boundary conditions on the Laplace transform $\overline{\psi}$ are

$$\overline{\psi}(0,z,p) = (u_0/kp)\sin kz, \tag{29}$$

and $\overline{\psi}(x,z,p) \to 0$ as $x \to \pm\infty$ for $\mathrm{Re}(p) > 0$.

From (26) the latter condition implies that

$$\mathrm{Re}(\alpha) > 0 \quad \text{for} \quad \mathrm{Re}(p) > 0 \quad \text{for} \quad x > 0, \tag{30}$$

and

$$\mathrm{Re}(\alpha) < 0 \quad \text{for} \quad \mathrm{Re}(p) > 0 \quad \text{for} \quad x < 0. \tag{31}$$

(i) *The upstream solution* ($x < 0$)

The only value of α satisfying (31) is α_4 and hence the solution of (25) satisfying (29) is

$$\overline{\psi}(x,z,p) = (u_0/kp)\exp(-\alpha_4 kx)\sin kz.$$

On using the Laplace inversion integral we obtain

$$\psi(x,z,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{u_0}{kp}\sin kz\exp(-\alpha_4 kx + pt)\,dp,$$

where γ is chosen to be greater than the real parts of the singularities of α_4. Since $\alpha_4 \sim -1$ as $p \to \infty$, we can complete the contour by a large semicircle in $\mathrm{Re}(p) < 0$ and contours enclosing the branch cuts of α_4. The contributions to ψ from the semicircle in $\mathrm{Re}(p) < 0$ will be zero, as the integrand is $O(e^{pt}/p)$ and the contributions from the contours enclosing the branch cuts of α_4 are

$$O(t^{-\frac{3}{2}}\sin kz\exp\{-(1-R_i^{-\frac{2}{3}})^{\frac{3}{2}}\,Ukt + (1-R_i^{-\frac{2}{3}})^{\frac{1}{2}}\,kx\}) \quad \text{for} \quad R_i > 1,$$

and

$$O(t^{-\frac{3}{2}}\sin kz\exp\{\pm i(R_i^{-\frac{2}{3}}-1)^{\frac{3}{2}}\,Ukt \pm i(R^{-\frac{2}{3}}-1)^{\frac{1}{2}}\,kx\}) \quad \text{for} \quad R_i < 1. \tag{32}$$

So in the limit as $t \to \infty$ the solution for ψ will be determined by the residue of $\overline{\psi}\,e^{pt}$ at $p = 0$ and hence in the limit $t = \infty$,

$$\left.\begin{aligned} \psi &= (u_0/k)\sin kz\exp(1-R_i^{-2})^{\frac{1}{2}}\,kx \quad &\text{for} \quad R_i > 1,\\ \psi &= (u_0/k)\sin kz &\text{for} \quad R_i < 1. \end{aligned}\right\} \tag{33}$$

Thus for $R_i > 1$, the perturbation flow decays exponentially as $x \to -\infty$ and has an irrotational character, reducing to the irrotational solution in the limit $R_i = \infty$; but for $R_i < 1$, the solution is independent of x and describes a one-dimensional flow extending to upstream infinity.

The corresponding upstream solution ($x < 0$) in the limit $t \to \infty$, when the perturbation

$$\psi(r, 0, t) = k^{-1} w_0 r J_1(kr) H(t)$$

is introduced on the plane $x = 0$ of a fluid rotating with angular velocity Ω and having uniform axial velocity U along Ox, is

$$\left.\begin{aligned} \psi &= k^{-1} w_0 r J_1(kr) \exp{(1-R_0^{-2})^{\frac{1}{2}} kx} \quad \text{for} \quad R_0 > 1, \\ \psi &= k^{-1} w_0 r J_1(kr) \quad \text{for} \quad R_0 < 1, \end{aligned}\right\} \tag{34}$$

where $R_0 = Uk/2\Omega$.

(ii) *The downstream solution* ($x > 0$)

The general solution of (25) satisfying the downstream boundary condition (30) is

$$\bar{\psi}(x, z, p) = k^{-1} \sin kz \,[A_1 \exp{(-\alpha_1 kx)} + A_2 \exp{(-\alpha_2 kx)} + A_3 \exp{(-\alpha_3 kx)}].$$

So to determine the downstream solution uniquely, three boundary conditions are required on the plane $x = 0$, whereas only one condition was necessary for the upstream solution. The author has so far been unable to discover whether this is a characteristic of all non-steady rotational flows, though the following argument may throw some light on the necessity for more boundary conditions to specify the downstream flow than the upstream flow. The velocity and position of all the fluid particles in the upstream flow are known at the initial instant and the problem is to find the effect of the perturbation on $x = 0$. However, the velocity and density of the fluid particles entering the downstream flow on $x = 0$ are not completely specified by one condition on the plane $x = 0$.

Let the perturbation introduced on the plane $x = 0$ at the time $t = 0$ in a uniform flow of stratified fluid be given by

$$\begin{aligned} u(0, z, t) &= u_0 \cos kz = p\bar{u}(0, z, p), \\ w(0, z, t) &= w_0 \sin kz = p\bar{w}(0, z, p), \\ \rho(0, z, t) &= \sigma_0 \sin kz = p\bar{\rho}(0, z, p). \end{aligned}$$

The above boundary conditions lead to the following relations between A_1, A_2 and A_3

$$A_1 + A_2 + A_3 = \frac{u_0}{p}, \quad \alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3 = \frac{w_0}{p}, \quad \sum_{i=1}^{3} \frac{\beta \alpha_i A_i}{(p - U\alpha_i k)} = \frac{\sigma_0}{p}.$$

These give rise to analytic solutions for A_1, A_2 and A_3 as functions of p except at the branch points of α_1, α_2 and α_3 and at the origin. The solution for ψ is

$$\psi(x, z, t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\sin kz}{k} \sum_{i=1}^{3} A_i \exp{(-\alpha_i kx + pt)}\, dp.$$

From figure 1 we have that

$$\alpha_1 \sim 1, \quad \alpha_2 \sim p/Uk, \quad \alpha_3 \sim p/Uk \quad \text{as} \quad p \to \infty,$$

so that for $Ut > x$ the contour can be completed by a large semicircle in $\operatorname{Re}(p) < 0$ and contours enclosing the branch cuts of α_1, α_2 and α_3. The contributions from the branch cuts in $\mathrm{Re}(p) > 0$ will be zero as only α_1, α_2 and α_3 have branch points in $\mathrm{Re}(p) > 0$ and from the symmetry of the above relations between the A_i, it follows that across a branch cut of α_1 and α_2 (say), $A_1 \exp(-\alpha_1 kx)$ will go into $A_2 \exp(-\alpha_2 kx)$. The branch points of α_1, α_2 and α_3 in $\mathrm{Re}(p) \leqslant 0$, which are also branch points of α_4 will contribute to ψ and their contribution will be of the same order as (32). Hence by evaluating the residues of the A_i at the pole $p = 0$, we obtain the following solution for ψ in the limit $t = \infty$:

$$\psi = \frac{\sin kz}{k}\left\{u_0 - \frac{w_0}{(1-R_i^{-2})^{\frac{1}{2}}}[1-\exp-(1-R_i^{-2})^{\frac{1}{2}}kx]\right\} \quad \text{for} \quad R_i > 1,$$

$$\psi = \frac{\sin kz}{k}\left\{u_0 \frac{[1+R_i\cos(R_i^{-2}-1)^{\frac{1}{2}}kx]}{(1+R_i)} - \frac{w_0 \sin(R_i^{-2}-1)^{\frac{1}{2}}kx}{(R_i^{-2}-1)^{\frac{1}{2}}} + \frac{Uk\sigma_0}{\beta}\frac{[1-\cos(R_i^{-2}-1)^{\frac{1}{2}}kx]}{(1+R_i)}\right\} \quad \text{for} \quad R_i < 1.$$

Again the solution has an irrotational character for $R_i > 1$, though it also contains terms independent of x, provided $u_0 \neq 0$. Also it is interesting to note that σ_0 does not appear in the solution for ψ for $R_i > 1$, so that any density fluctuation with wave-number $k > \hat{g}/U$ does not affect the steady-state flow. Similarly, in rotating fluids any perturbation in the azimuthal velocity component v with wave-number $k > 2\Omega/U$ does not affect the u and w components of velocity as $t \to \infty$.

For flows in which $R_i < 1$ and also for $R_0 < 1$ terms giving rise to waves appear in the downstream solution as well as terms describing a one-dimensional or geostrophic flow.

6. Applications

(i) *Stratified flow into a line sink*

This problem has interesting practical applications to canals closed by gates, which do not fit closely at the bottom (see Debler 1961). Yih (1958) gives a solution for the steady flow of a stably stratified fluid between two horizontal boundaries into a line sink. His axes are chosen so that the boundaries are $z = 0$, $z = d$ and the line sink is at $z = 0$, $x = 0$. He assumes that the flow far upstream is such that $\rho U^2 = A^2$, where A is a constant and $\rho = \rho_0 - \beta z$, in which case the steady state is given by the solution of

$$\nabla^2\psi' - \frac{g\beta}{A\rho_0}z = -\frac{g\beta}{A^2\rho_0}\psi', \quad \left(\frac{\rho}{\rho_0}\right)^{\frac{1}{2}} u = \frac{\partial\psi'}{\partial z}, \quad \left(\frac{\rho}{\rho_0}\right)^{\frac{1}{2}} w = -\frac{\partial\psi'}{\partial x}, \tag{35}$$

subject to the following boundary conditions

$$\psi'(x,0) = 0, \quad \psi'(x,d) = Ad,$$

and

$$\psi'(0,z) = Ad \quad \text{for} \quad 0 < z \leqslant d.$$

Yih's solution is

$$\psi' = Az + \frac{2Ad}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\exp(n^2\pi^2 - F^{-2})^{\frac{1}{2}}\frac{x}{d}\sin\frac{n\pi z}{d},$$

where $F = A/d(g\beta)^{\frac{1}{2}}$ is the Froude number of the flow. Yih only claims validity for his solution in the range $F > \pi^{-1}$, as the solution for $F < \pi^{-1}$ can involve wave-like terms which do not occur in practice.

It is of interest to solve the problem of the flow into a line sink using the linearized equation (24). The linearization is obviously invalid in the neighbourhood of the sink, but it is assumed that further upstream the perturbation to the uniform flow is small. The relevant boundary conditions on the perturbation stream function ψ for $t > 0$ are

$$\psi(x,d,t) = \psi(x,0,t) = 0,$$

$$\psi(0,z,t) = U(d-z) = \frac{2Ud}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi z}{d} \quad (0 < z \leqslant d).$$

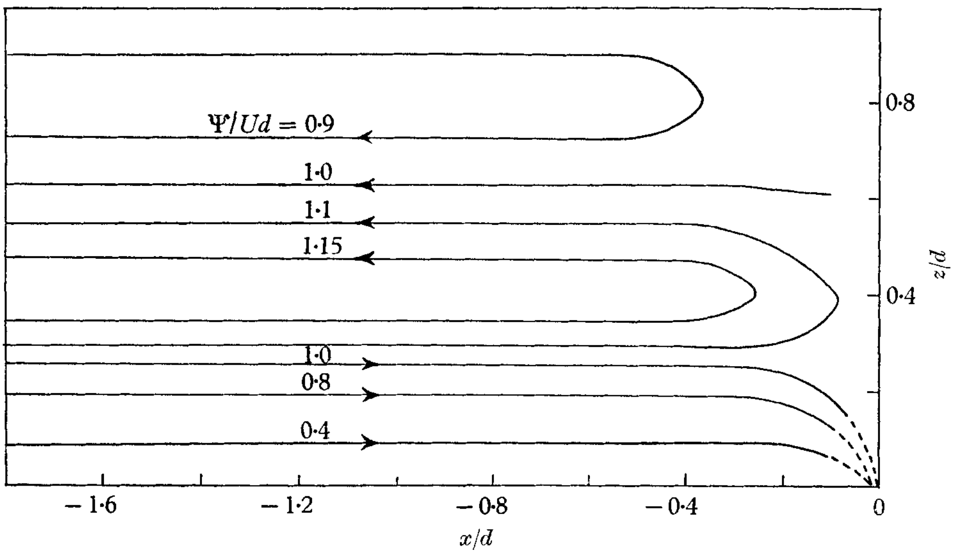

FIGURE 2. Streamlines for flow into a line sink for $F = \frac{2}{5}\pi$.

Using the analysis of § 5 and the results (33), we obtain the solution for $\Psi = \psi + Uz$, the total stream function of the flow in the limit $t = \infty$:

$$\Psi = Uz + \frac{2Ud}{\pi}\left\{\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi z}{d} + \sum_{n=N+1}^{\infty}\frac{1}{n}\sin\frac{n\pi z}{d}\exp\,(n^2\pi^2 - F^{-2})^{\frac{1}{2}}\frac{x}{d}\right\}, \quad (36)$$

for $x < 0$, where $N\pi \leqslant F^{-1} < (N+1)\pi$ and $F = U/d\hat{g}^{\frac{1}{2}}$.

It is seen that this solution is similar to Yih's solution for $F > \pi^{-1}$, but for $F \leqslant \pi^{-1}$ it contains terms which are independent of x and provides an explanation of the blocking of the upstream flow found experimentally by Debler (1961). In figure 2 the streamlines for $F = \frac{2}{5}\pi$ are sketched and it bears a qualitative resemblance to the flow observed by Debler, though he did not observe the alternate jets appearing in the blocked flow. This is possibly because the velocities in the blocked flow region are small. However the more probable reason is that for some ranges of z in the region, the flow is unstable in the sense that $\partial\rho/\partial z > 0$ and that this leads to stagnation zones. It is easy to show that the number of jets occurring in the blocked flow is $N+1$ for Froude numbers satisfying $N\pi \leqslant F^{-1} < (N+1)\pi$.

However the assumption underlying the theory, that the perturbation to the upstream flow remains small, must be wrong for $F < \pi^{-1}$ as the solution gives a finite upstream perturbation. Hence the assumption of a uniform undisturbed upstream flow, which has been basic to most theories in both stratified flow and rotating fluids is probably not valid. This implies that the steady-state equation, which takes a linear form in both theories with the aforementioned assumption, becomes more difficult to use as it seems impossible to see what assumptions can be made about the upstream flow.

(ii) *Upstream flow due to a porous disk in a rotating fluid*

Unfortunately equation (13) and the results of § 5 cannot be used to determine the steady flow past a disk or solid of revolution, which is moved with uniform velocity along the axis of a rotating fluid, as the linearization of the non-linear inertia terms is invalid in the neighbourhood of the body. Even if the approximation could be justified, it has so far proved impossible to formulate the boundary conditions on the plane $x = 0$ in such a way that the problem can be solved. However the nature of the upstream flow due to such a body is probably not very different from the solution of the following problem.

We consider the upstream flow produced by the introduction of the following perturbation on $x = 0$ for $t \geqslant 0$

$$w = -w_0 \qquad \text{for} \quad r < a,$$
$$w = \frac{2w_0}{\pi}\left\{\frac{a}{(r^2-a^2)^{\frac{1}{2}}} - \sin^{-1}\frac{a}{r}\right\} \quad \text{for} \quad r > a,$$

which can be expressed alternatively as

$$w = \frac{2w_0}{\pi}\int_0^\infty J_0(kr)\left(a\cos ka - \frac{\sin ka}{k}\right)dk.$$

This perturbation is suggested by the results of Morgan (1951) and Stewartson (1952) for a disk and sphere respectively, which are moved slowly along the axis of a rotating fluid.

Using the analysis of § 5 and the results (34), we obtain the following solution for w in the limit as $t \to \infty$

$$w = \frac{2w_0}{\pi}\left\{\int_0^{2\Omega/U} J_0(kr)\left(a\cos ka - \frac{\sin ka}{k}\right)dk \right.$$
$$\left. + \int_{2\Omega/U}^\infty J_0(kr)\left(a\cos ka - \frac{\sin ka}{k}\right)\exp\left[(k^2a^2 - S^{-2})^{\frac{1}{2}}\frac{x}{a}\right]dk\right\} \qquad (37)$$

where $S = U/2\Omega a$ is the Rossby number of the flow.

Since the solutions of Morgan and Stewartson predicted infinite velocities on the circumscribing cylinder $r = a$, it is of interest to enquire whether the velocities predicted by the above solution remain finite in the limit as $S \to 0$. After some algebra the following solution is obtained for w

$$w = -w_0 + O(S^{\frac{1}{2}}) \quad \text{for} \quad r < a,$$
$$w = \tfrac{1}{2}\pi^{-\frac{3}{2}}w_0 S^{-\frac{1}{2}} + O(1) \quad \text{for} \quad r = a,$$
$$w = \frac{2w_0}{\pi}\left\{\frac{a}{(r^2-a^2)^{\frac{1}{2}}} - \sin^{-1}\frac{a}{r}\right\} + O(S^{\frac{1}{2}}) \quad \text{for} \quad r > a.$$

This solution is identical with that of Stewartson and Morgan except on $r = a$. The present result gives velocities $O(w_0 S^{-\frac{1}{2}})$ on the circumscribing cylinder, which suggests that in the problems considered by Morgan and Stewartson the velocities on $r = a$ are $O(U\Omega a)^{\frac{1}{2}}$. So in the limit as $U \to 0$ the velocities on $r = a$ would tend to zero, but more slowly than elsewhere in the fluid, where the velocities are $O(U)$. It can easily be shown that the width of the shear layer about $r = a$, in which the velocities change rapidly is $O(aU/\Omega)^{\frac{1}{2}}$. Morrison & Morgan (1956) (see table 1, case (ii)) found that the velocity in the shear layer is $O(U(\nu x/2\Omega a^3)^{-\frac{1}{3}})$ and that the width of the shear layer is $O(\nu x/2\Omega)^{\frac{1}{3}}$, both quantities depending on x.

It is worth noting that (37) reduces to the irrotational solution for the upstream flow past a porous disk in the limit of infinite Rossby number (see Lamb 1932, p. 138).

7. The use of source distributions for generating stratified fluid flow over a barrier

Yih (1960) gives a method of generating solutions for the flow between parallel planes $z = 0$ and $z = d$ over a barrier of unspecified form, using the steady-state equation (35). This problem has a bearing on the lee-waves which are observed on the lee side of mountain ridges. For a given source of vorticity distribution $f(z)$ on $x = 0$, Yih determines the upstream and downstream solutions by assuming the non-existence of upstream waves and matching the solutions across $x = 0$. Yih finds that the amplitudes of the lee-waves are not dependent on the exact form of $f(z)$, but only on certain of its integral properties. His solution contains no terms independent of x, and so does not account for the blocking observed by Long (1955) and Debler (1961).

We now consider the problem in which an irrotational source distribution is switched on in the plane $x = 0$ at the time $t = 0$ and maintained there. The flow originally consists of a uniform stream of stably stratified fluid flowing with uniform velocity U between the planes $z = 0$ and $z = d$. The perturbation is assumed to be small so that the linearized equations (19) and (20) can be used. For a source distribution on $x = 0$ of strength $f(z)$, the continuity equation (21) becomes

$$\partial u/\partial x + \partial w/\partial z = \delta(x) f(z) H(t). \tag{38}$$

From (38) a modified perturbation stream function ψ can be defined by

$$u = \partial\psi/\partial z, \quad w = -\partial\psi/\partial x + \delta(x) g(z) H(t), \quad g'(z) = f(z). \tag{39}$$

On eliminating p and ρ from (19) and (20) we obtain

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = \hat{g}\,\frac{\partial w}{\partial x},$$

and on taking the Laplace transform as defined by (6), the above equation becomes

$$\left(p + U\frac{\partial}{\partial x}\right)^2 \left(\frac{\partial \overline{u}}{\partial z} - \frac{\partial \overline{w}}{\partial x}\right) - \hat{g}\,\frac{\partial \overline{w}}{\partial x} = \left(p + 2U\frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)_{t=0} + \left(\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\right)_{t=0}. \tag{40}$$

From (19) we have that

$$\rho_0 \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = g\frac{\partial \rho}{\partial x},$$

which shows that if the initial density perturbation is zero everywhere, then the initial rate of change of vorticity is zero, since the initial perturbation flow is irrotational. So the right-hand side of (40) is zero and using (39), (40) reduces to

$$\left(p + U\frac{\partial}{\partial x}\right)^2 \left(\nabla^2\overline{\psi} - \frac{\delta'(x)g(z)}{p}\right) + \hat{g}\frac{\partial^2\overline{\psi}}{\partial x^2} - \hat{g}\frac{\delta'(x)g(z)}{p} = 0.$$

We now introduce the generalized Fourier transform as defined in Lighthill (1960) (here y replaces the $2\pi y$ of Lighthill's definition):

$$\overline{\psi}'(y,z,p) = \int_{-\infty}^{\infty} \overline{\psi}(x,z,p)\, e^{-iyx}\, dx.$$

On taking the Fourier transform of the above equation we obtain

$$\frac{d^2\overline{\psi}'}{dz^2} - \left(y^2 + \frac{\hat{g}y^2}{(p+Uiy)^2}\right)\overline{\psi}' = \left(iy + \frac{i\hat{g}y}{(p+Uiy)^2}\right)\frac{g(z)}{p}.$$

The boundary conditions on $z = 0$ and $z = d$ require that

$$\overline{\psi}'(y,0,p) = \overline{\psi}'(y,d,p) = 0.$$

Using the method of variation of parameters, we obtain the solution satisfying the above boundary conditions,

$$\overline{\psi}' = \frac{\gamma}{iyp}\left\{\frac{\sin\gamma(z-d)}{\sin\gamma d}\int_0^z g(\tau)\sin\gamma\tau\, d\tau + \frac{\sin\gamma z}{\sin\gamma d}\int_z^d g(\tau)\sin\gamma(\tau-d)\, d\tau\right\},$$

where

$$\gamma^2 = -y^2 - \hat{g}y^2/(p+Uiy)^2.$$

The generalized Fourier inversion integral is

$$\overline{\psi}(x,z,p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\psi}'(y,z,p)\, e^{ixy}\, dy,$$

and for $x \gtrless 0$ the contour can be completed by a large semicircle in $\operatorname{Im}(y) \gtrless 0$ respectively, along which the integrand is exponentially small. The integrand has poles at the values of y given by the roots of

$$y^2 + \hat{g}y^2/(p+Uiy)^2 = -n^2\pi^2/d^2 \quad (n = 1, 2, 3, \ldots). \tag{41}$$

On writing $y = i\alpha n\pi/d$, $s = \hat{g}^{-\frac{1}{2}}p$ and $R_{in} = d^{-1}\hat{g}^{-\frac{1}{2}}Un\pi$, (41) reduces to

$$(s - R_{in}\alpha)^2(\alpha^2 - 1) + \alpha^2 = 0,$$

which is identical with (28), and so the roots of (41) are

$$y_{nj} = i\alpha_j n\pi/d \quad (j = 1, 2, 3, 4),$$

where $\alpha_j(p)$ is defined as a single-valued function of p as in figure 1.

It should be noticed that the integrand is a single-valued function of y although $\sin\gamma d$ is not. Now

$$\mathrm{Re}\,(\alpha_i) = \frac{d}{n\pi}\,\mathrm{Im}\,(y_{ni}) > 0 \quad \text{for} \quad \mathrm{Re}\,(p) > 0 \quad (i = 1, 2, 3),$$

and
$$\mathrm{Re}\,(\alpha_4) = \frac{d}{n\pi}\,\mathrm{Im}\,(y_{n4}) < 0 \quad \text{for} \quad \mathrm{Re}\,(p) > 0,$$

and so as expected only α_4 contributes to the flow for $x < 0$, whereas α_1, α_2 and α_3 describe the flow for $x > 0$. After some manipulation the following solution is obtained for $\bar{\psi}$:

$$\bar{\psi} = -\sum_{n=1}^{\infty} \frac{(p - U\alpha_4 n\pi/d)}{pd(p - U\alpha_4^3 n\pi/d)} \times \exp\left(-\alpha_4 \frac{n\pi x}{d}\right) \sin\frac{n\pi z}{d} \int_0^d \sin\frac{n\pi\tau}{d}\, g(\tau)\, d\tau \quad \text{for} \quad x < 0,$$

and
$$\bar{\psi} = \sum_{n=1}^{\infty}\sum_{i=1}^{3} \frac{(p - U\alpha_i n\pi/d)}{pd(p - U\alpha_i^3 n\pi/d)} \times \exp\left(-\alpha_i \frac{n\pi x}{d}\right) \sin\frac{n\pi z}{d} \int_0^d \sin\frac{n\pi\tau}{d}\, g(\tau)\, d\tau \quad \text{for} \quad x > 0.$$

Now $\bar{\psi}$ has a pole at $p = 0$ and branch points at the branch points of the α_i, since $p - U\alpha_i^3 n\pi/d$ vanishes only at the branch points of the α_i (see § 4). The contributions to ψ from the branch cuts are algebraically or exponentially small as $t \to \infty$, the branch cuts in $\mathrm{Re}\,(p) > 0$ having zero contribution as in § 5. After further manipulation we obtain the following solution for ψ in the limit as $t \to \infty$, using the values of α_i at $p = 0$ as defined in figure 1.

$$\psi = \sum_{n=1}^{N} \frac{1}{(n\pi F - 1)\,d} \sin\frac{n\pi z}{d} \int_0^d g(\tau) \sin\frac{n\pi\tau}{d}\, d\tau - \sum_{n=N+1}^{\infty} \frac{n^2\pi^2F^2}{(n^2\pi^2F^2 - 1)} \exp\,(n^2\pi^2 - F^{-2})^{\frac{1}{2}}\frac{x}{d} \sin\frac{n\pi z}{d} \int_0^d g(\tau) \sin\frac{n\pi\tau}{d}\, d\tau \quad \text{for} \quad x < 0,$$

and

$$\psi = \sum_{n=1}^{N} \left\{\frac{1}{(n\pi F + 1)\,d} + \frac{2n^2\pi^2F^2}{(n^2\pi^2F^2 - 1)\,d} \cos\,(F^{-2} - n^2\pi^2)^{\frac{1}{2}} \frac{x}{d}\right\} \sin\frac{n\pi z}{d} \int_0^d \sin\frac{n\pi\tau}{d}\, g(\tau)\, d\tau + \sum_{n=N+1}^{\infty} \left\{\frac{2}{(1 - n^2\pi^2F^2)\,d} + \frac{n^2\pi^2F^2}{(n^2\pi^2F^2 - 1)\,d} \exp - (n^2\pi^2 - F^{-2})^{\frac{1}{2}} \frac{x}{d}\right\} \times \sin\frac{n\pi z}{d} \int_0^d \sin\frac{n\pi\tau}{d}\, g(\tau)\, d\tau \quad \text{for} \quad x > 0, \tag{42}$$

where $R_{in} = n\pi F$ and $N\pi < F^{-1} < (N+1)\pi$.

The solution (42) is different from that obtained by Yih who uses a different type of source distribution, though the property that the amplitude of the lee-waves is not dependent on the exact form of $f(z)$ is retained. The main difference in the solutions is the presence of terms independent of x in (42). These are the terms which give rise to the jets upstream of the obstacle, observed by Long (1955) for low Froude numbers. Yih eliminates the possibility of obtaining such terms by his assumption of uniform upstream conditions.

For values of $F^{-1} = m\pi$, where m is an integer, the solution for ψ is singular and resonance occurs. The solution for a source and a dipole with its axis along Oz at $x = 0$, $z = z_0$, can be obtained by putting $f(z) = \delta(z-z_0)$ and $\delta'(z-z_0)$ respectively. It is interesting to note that the solution for a dipole with its axis along the direction of the uniform stream, obtained by differentiating the solution for a source with respect to x, has no terms independent of x and so its influence does not extend to upstream infinity. However the solution for a dipole with its axis perpendicular to the uniform stream does contain terms independent of x. This is to be expected when one considers that the effects of stratification are to inhibit motions in the Oz direction.

8. The ring source in a rotating fluid

The general results for sources in a stratified fluid can easily be extended to rotating fluids by replacing the trigonometric functions with Bessel functions. However it is of more interest to solve one of the problems considered by Fraenkel (1956) and to compare the results.

Fraenkel solves the problem for the steady flow of a rotating fluid through a pipe and past a ring source on the wall of the pipe. The axis Ox is chosen so as to coincide with the axis of the pipe, which is of unit radius, and the ring source is situated on the pipe wall at $x = 0$. The boundary conditions on the perturbation stream function ψ are

$$\psi(1,x) = 0, \quad x < 0; \qquad \psi(1,x) = m, \quad x > 0; \qquad \psi(0,x) = 0.$$

Fraenkel assumes that the flow far upstream consists of a solid body rotation with angular velocity Ω about Ox and a uniform axial velocity U. The equation satisfied by ψ reduces to

$$r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{\partial^2\psi}{\partial x^2}+\frac{4\Omega^2}{U^2}\psi = 0 \tag{43}$$

(see Squire 1956, p. 156). Fraenkel obtains the following solution:

$$\psi(r,x) = -mr\sum_{n=N+1}^{\infty}\frac{\exp\left((j_n^2-S^{-2})^{\frac{1}{2}}x\right)j_n J_1(j_n r)}{(j_n^2-S^{-2})\,J_0(j_n)} \quad \text{for} \quad x < 0,$$

and

$$\psi(r,x) = mr\left\{\sum_{n=1}^{N}\frac{2\cos\left[(S^{-2}-j_n^2)^{\frac{1}{2}}x\right]j_n J_1(j_n r)}{(j_n^2-S^{-2})\,J_0(j_n)} + \sum_{n=N+1}^{\infty}\frac{\exp\left[-(j_n^2-S^{-2})^{\frac{1}{2}}x\right]j_n J_1(j_n r)}{(j_n^2-S^{-2})\,J_0(j_n)}+\frac{J_1(S^{-1}r)}{J_1(S^{-1})}\right\} \quad \text{for} \quad x > 0, \tag{44}$$

where j_n is the nth non-zero root of $J_1(x) = 0$, $S = U/2\Omega$ is the Rossby number of the flow and $j_N < S^{-1} < j_{N+1}$.

We now consider the problem in which the ring source is switched on at $t = 0$. This problem is ideal for solution by the linearized equation as the strength of the source can be made arbitrarily small. Using the same method as in § 7, we obtain the following solution for ψ in the limit $t\to\infty$.

$$\psi = mr\left\{\sum_{n=1}^{N}\frac{1}{(Sj_n-1)}\frac{J_1(j_n r)}{j_n J_0(j_n)} - \sum_{n=N+1}^{\infty}\frac{\exp\left[(j_n^2-S^{-2})^{\frac{1}{2}}x\right]j_n J_1(j_n r)}{(j_n^2-S^{-2})\,J_0(j_n)}\right\} \quad \text{for} \quad x < 0,$$

and

$$\psi = mr\left\{\sum_{n=1}^{N}\left\{\frac{1}{(Sj_n+1)}+\frac{2j_n^2\cos(S^{-2}-j_n^2)^{\frac{1}{2}}x}{(j_n^2-S^{-2})}\right\}\frac{J_1(j_nr)}{j_nJ_0(j_n)}\right.$$
$$\left.+\sum_{n=N+1}^{\infty}\left\{\frac{2}{(1-S^2j_n^2)}+\frac{j_n^2\exp(-(j_n^2-S^{-2})^{\frac{1}{2}}x)}{(j_n^2-S^{-2})}\right\}\frac{J_1(j_nr)}{j_nJ_0(j_n)}-2\sum_{n=1}^{\infty}\frac{J_1(j_nr)}{j_nJ_0(j_n)}\right\}$$
$$\text{for}\quad x>0\quad\text{and}\quad 0\leqslant r<1, \qquad (45)$$

with the same meaning as above for S, j_n and N.

On comparing the solutions (44) and (45) we see that the exponential and wave terms are identical, but that (45) contains additional geostrophic terms. If we put $U = 0$ in (45) we obtain the solution appropriate to the neglect of the non-linear inertia terms, namely

$$\psi = -mr\sum_{n=1}^{\infty}J_1(j_nr)/\{j_nJ_0(j_n)\}\quad\text{for}\quad x<0,$$
$$\psi = -mr\sum_{n=1}^{\infty}J_1(j_nr)/\{j_nJ_0(j_n)\} = \tfrac{1}{2}mr^2\quad\text{for}\quad x>0\quad\text{and}\quad 0\leqslant r<1.$$

Thus the following conclusions are suggested by (45). First, Taylor (1922), Long (1953*a*) and Fraenkel (1956) found no geostrophic terms in their solutions because of their assumption of zero perturbation to the upstream flow at infinity, which led to equation (43) for the perturbation stream function. Secondly, Morgan (1951) and Stewartson (1952) found no waves or irrotational-like terms in their solutions, as these terms arise from the non-linear inertia terms, which are neglected in their equations of motion.

The steady-state problem of the ring source considered by Fraenkel could also be set up in the following way. Consider the problem in which a ring source and a ring sink are switched on at $t = 0$, on the pipe wall at $x = 0$. During the subsequent flow the ring sink is moved downstream with constant velocity V so that in the limit as $t\to\infty$, we are left with only a ring source at $x = 0$. The solution for ψ for this problem in the limit $t = \infty$ is

$$\psi = mr\left\{\sum_{n=1}^{N}\frac{1}{(Sj_n-1)(j_n-2\Omega V^{-1}(Sj_n-1))}\frac{J_1(j_nr)}{J_0(j_n)}\right.$$
$$\left.-\sum_{n=N+1}^{\infty}\frac{\exp(j_n^2-S^{-2})^{\frac{1}{2}}x}{(j_n^2-S^{-2})}\frac{j_nJ_1(j_nr)}{J_0(j_n)}\right\}\quad\text{for}\quad x<0,$$

and

$$\psi = mr\left\{\sum_{n=1}^{N}\left(\frac{j_n}{(Sj_n+1)(j_n-2\Omega V^{-1}(Sj_n+1))}+\frac{2j_n^2\cos(S^{-2}-j_n^2)^{\frac{1}{2}}x}{(j_n^2-S^{-2})}\right)\frac{J_1(j_nr)}{J_0(j_n)}\right.$$
$$+\sum_{n=N+1}^{\infty}\left(\frac{j_n(8\Omega V^{-1}Sj_n-2j_n)}{(S^2j_n^2-1)[(j_n-2\Omega V^{-1}Sj_n)^2-4\Omega^2V^{-2}]}+\frac{j_n^2\exp[-(j_n^2-S^{-2})^{\frac{1}{2}}x]}{(j_n^2-S^{-2})}\right)$$
$$\left.\times\frac{J_1(j_nr)}{J_0(j_n)}-2\sum_{n=1}^{\infty}\frac{J_1(j_nr)}{j_nJ_0(j_n)}\right\}\quad\text{for}\quad x>0\quad\text{and}\quad 0\leqslant r<1. \qquad (46)$$

On comparing (45) and (46) it is seen that only the geostrophic terms are different. This result suggests the following important hypothesis, namely, that the wave-like and irrotational-like terms are determined by the geometry of the problem and are independent of the way the steady state is achieved, whereas the geostrophic terms depend on the means of setting up the steady motion as well as on the geometry of the problem.

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